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Upstream propagation of unsteady disturbances in supersonic boundary layers

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It is shown that weak harmonic waves can propagate upstream in a supersonic stream along a plane wall over distances that are very large in comparison with the boundary-layer thickness. These waves are characterized by small reduced frequencies, large wavelengths and small phase velocities. Results are presented for the flow along a single plane wall and also for the flow between two plane parallel plates (channel flow).

The progressive-wave solutions are found by asymptotic expansions for small disturbances, large Reynolds numbers of the basic flow, and small reduced frequencies of the unsteady disturbances. It turns out that, as in the corresponding steady theory, four flow regions have to be distinguished: a middle layer which embraces most of the boundary layer; an inner layer near the wall; the outer flow field; and a transition layer between the middle layer and the outer layer. A quasi-steady treatment of the middle, transitional and outer layers is appropriate. Unsteady effects originate in the inner layer.

The relative importance of viscosity and unsteady effects with regard to the waves is characterized by a dimensionless parameter N which is the product of certain powers of the reduced frequency and the Reynolds number. For $N \to \infty$ and a single wall, Lighthill's steady theory of upstream influence in supersonic boundary layers is recovered as a limiting case of the present theory.

1. Introduction

We shall be concerned in this paper with small disturbances propagating upstream in a gas that flows along a plane wall at a supersonic speed. Although the inviscid supersonic flow is governed by hyperbolic equations and the thin boundary layer near the wall approximately obeys parabolic equations it is well known from many experiments that small disturbances of the basic flow do have a noticeable upstream influence.

As far as steady disturbances are concerned, theories which are able to predict correctly the experimental findings were first developed by Müller (1953, 1955) and by Lighthill (1953). Considerable advances have been made since then in the steady theory.

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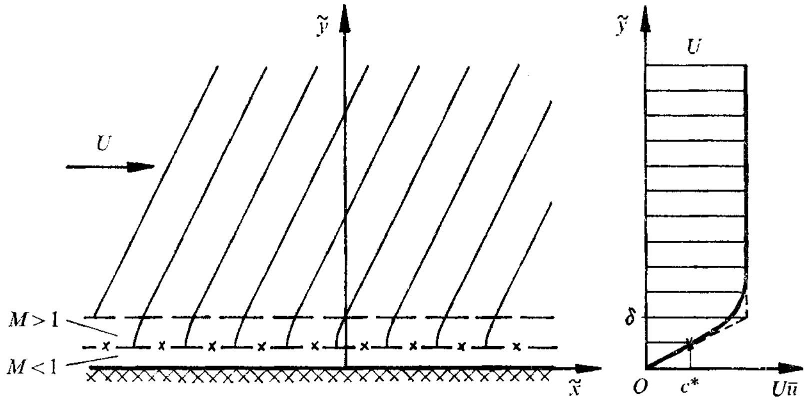

FIGURE 1. Basic (undisturbed) shear flow at a plane wall.

In the present paper, however, we are concerned with unsteady phenomena. Following Lighthill's approach, Dore (1967) determined the upstream influence of a weak shock that moves with a small uniform velocity parallel to the wall. The main result is that the upstream influence is greater than or less than that for the steady case according as the shock velocity is in or against the direction of the main stream. Dore considers his work to be a first step towards an understanding of the unsteady interaction effects which are of great practical importance when the shock wave oscillates about some mean position.

In what follows, a theory for the upstream propagation of unsteady disturbances with particular consideration of harmonic waves will be given. For a single wall, Lighthill's theory will be recovered as a certain limiting case of the present, more general theory. In addition the problem of disturbances propagating upstream in a channel with plane parallel walls will be considered. The presence of the second wall causes some peculiarities with respect to unsteady disturbances as well as to steady ones.

The basic (undisturbed) flow is considered to be a plane steady shear flow along a plane wall, with the free-stream velocity U and the velocity profile $U\bar{u}$ assumed to be independent of the tangential co-ordinate $\tilde{x}$ (figure 1). This approximation rests on the fact that for very large Reynolds numbers the actual boundary-layer flow varies only slowly with increasing $\tilde{x}$.

At a certain distance from the wall the velocity of the shear flow is equal to the critical sound velocity c^*. Hence the boundary-layer-like shear flow is divided by a sonic line into two distinct regions (figure 1): above the sonic line the flow is supersonic ($M > 1$), below the sonic line the flow is subsonic ($M < 1$). In the subsonic region the upstream propagation of small disturbances is possible, at least in principle. However, a perturbation of the subsonic flow will cause also a perturbation of the supersonic flow above. Hence a wave propagating upstream in the subsonic region loses energy and is damped accordingly. The question is now whether unsteady disturbances can propagate far upstream (far in comparison with the boundary-layer thickness) before they are virtually extinguished owing to the damping.

The answer will of course depend on the frequency ω of the disturbances, or more specifically, on a 'reduced' (dimensionless) frequency Ω defined by

$$\Omega = \omega\delta/U, \tag{1.1}$$

where for later convenience the 'boundary-layer thickness' δ is defined by

$$\delta = (d\bar{u}/d\tilde{y})^{-1}_{\tilde{y}=0}, \tag{1.2}$$

cf. figure 1.

It might be tempting to think that an ordinary sound wave, when emitted from a point in the subsonic layer, could be transmitted quite far along a ray pointing upstream. However, that the sound wave has to propagate in a highly rotational shear flow has to be taken into account. The concept of a ray applies if the wavelength of the sound wave is very small in comparison with the boundary-layer thickness δ, i.e. for large values of the reduced frequency Ω. But it is known from geometric acoustics that the ray curvature is of the same order of magnitude as the vorticity of the shear flow divided by the local sound velocity (Landau & Lifshitz 1959). Hence in our problem the radius of curvature of the ray will be of the order of the boundary-layer thickness. Therefore an acoustic disturbance propagating along a ray that initially points upstream will have already left the subsonic region after a path length of the order of the boundary-layer thickness. Since in the supersonic region the acoustic disturbance will, of course, drift downstream, the distance over which disturbances of large reduced frequencies are felt upstream is of the order of the boundary-layer thickness, i.e. relatively small.

Large reduced frequencies are therefore ruled out from our considerations, and in what follows we shall focus our attention on small reduced frequencies. Since in the latter case the characteristic length of the disturbances (the wavelength) is large in comparison with the characteristic length of the basic shear flow (the boundary-layer thickness), geometric acoustics do not apply. Hence it is not unlikely that unsteady disturbances of small reduced frequencies (unlike those of large reduced frequencies) can propagate far upstream. The investigation presented in this paper will show that this is in fact the case.

Apart from this consideration we should mention that the case of small reduced frequencies Ω is also the most interesting one from a practical point of view since, according to (1.1), the actual frequency ω is to be multiplied by δ/U, which is a very small time for typical supersonic boundary layers.

It is important to note that very small reduced frequencies do not necessarily lead to quasi-steady behaviour of the disturbances. Near the wall the velocity of the steady shear flow is very small and the unsteady terms can be essential there no matter how small Ω.

As far as viscosity effects are concerned it will be advantageous to use the Reynolds number

$$Re = \delta U \rho_w/\mu_w, \tag{1.3}$$

which is based on the boundary-layer thickness δ and on the wall values ρ_w and μ_w of the density and viscosity. We shall of course assume the Reynolds number to be large. The investigation will, strictly speaking, be restricted to laminar flow. However, the results can also be employed for a turbulent boundary layer (with a laminar sublayer) if we accept the hypothesis (Inger & Williams 1972; McClure 1962) that there is no correlation between the turbulent fluctuations and the unsteady disturbances whose upstream propagation is to be studied. This permits us to treat the turbulent shear flow as a quasi-laminar mean flow controlled by an appropriate eddy viscosity.

2. Basic equations

Dimensionless variables are now introduced. The Cartesian co-ordinates x and y are based on the boundary-layer thickness δ, which is defined in (1.2), i.e. $x = \tilde{x}/\delta$ and $y = \tilde{y}/\delta$. The velocity components u and v (tangential and normal to the wall, respectively) are based on the free-stream velocity U, and the time t on a characteristic time scale ω^{-1} of the unsteady disturbances, ω being the frequency in the case of periodic disturbances. The pressure p is made dimensionless by $\rho_w U^2$, where ρ_w is the value of the density of the basic shear flow at the wall. All other thermodynamic quantities such as the density ρ, temperature T, etc., as well as the viscosity μ, second viscosity μ' and heat conductivity λ_c are based on their respective wall values (subscript w).

Then for plane laminar flow of a perfect gas the conservation equations, written in dimensionless variables, are the equation of continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.1}$$

the Navier–Stokes equations

$$\rho\frac{Du}{Dt}+\frac{\partial p}{\partial x} = \frac{1}{Re}\left\{2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\mu_w'}{\mu_w}\frac{\partial}{\partial x}\left[\mu'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\right\}, \tag{2.2}$$

$$\rho\frac{Dv}{Dt}+\frac{\partial p}{\partial y} = \frac{1}{Re}\left\{2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\mu_w'}{\mu_w}\frac{\partial}{\partial y}\left[\mu'\left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)\right]\right\} \tag{2.3}$$

and the energy equation

$$\frac{1}{p}\frac{Dp}{Dt}-\frac{\gamma}{\rho}\frac{D\rho}{Dt} = \frac{1}{Re\,p}\left\{(\gamma-1)\,\Phi+\frac{1}{Pr\,M_w^2}\left[\frac{\partial}{\partial x}\left(\lambda_c\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_c\frac{\partial T}{\partial y}\right)\right]\right\}. \tag{2.4}$$

Here D/Dt indicates the substantial derivative, i.e.

$$\frac{D}{Dt} = \Omega\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}, \tag{2.5}$$

with Ω given in (1.1). Pr is the constant Prandtl number, γ the ratio of the specific heats at constant pressure and constant volume, M_w a Mach number which is defined as the ratio of the free-stream velocity and the wall value of the sound velocity, and Φ is the dissipation;

$$\Phi = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\frac{\mu_w'}{\mu_w}\mu'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2. \tag{2.6}$$

The set of equations is to be completed by the equation of state of a perfect gas and by relations describing the temperature dependence of the viscosity and heat conductivity.

The boundary conditions at the wall, again in dimensionless notation, are

$$u = v = 0, \quad T = 1 \quad \text{at} \quad y = 0. \tag{2.7}$$

Further boundary conditions, for instance those which follow from relations of symmetry, will be introduced when needed.

3. Asymptotic expansions

According to the arguments given in the introduction we shall seek asymptotic solutions for small disturbances, small reduced frequencies and large Reynolds numbers, i.e. for

$$\epsilon, \Omega, Re^{-1} \rightarrow 0, \tag{3.1}$$

where the perturbation parameter ϵ is such that it characterizes the order of magnitude of the tangential velocity disturbances relative to the free-stream velocity U. The Mach number and Prandtl number are assumed to be $O(1)$. Here and in what follows the order symbol, say $\Omega = O(\epsilon)$, is used in the restricted sense indicating that $\Omega/\epsilon \rightarrow C$ as $\epsilon \rightarrow 0$, where C is not only bounded but also non-zero.

As the reduced frequency Ω goes to zero, the wavelength of the disturbances is supposed to go to infinity. This means that the wavenumber (or, in the case of a complex wavenumber, its real part) is supposed to go to zero as the reduced frequency Ω goes to zero. It might be tempting to assume that the wavenumber is of the same order of magnitude as Ω, since such an assumption would yield a propagation speed of order one, i.e. of the order of the sound speed when expressed in dimensional form. However, this is a wrong approach which does not lead to upstream-propagating waves.

We therefore introduce a parameter K such that the real part of the wavenumber is $O(K)$ but we do not commit ourselves *a priori* to a fixed order of magnitude for K in terms of Ω. We only require K to go more slowly to zero than Ω, i.e.

$$K \rightarrow 0, \quad \Omega/K \rightarrow 0 \quad \text{as} \quad \Omega \rightarrow 0, \tag{3.2}$$

since the other two possibilities ($\Omega/K = O(1)$ and $\Omega/K \rightarrow \infty$) do not lead to upstream-propagating waves and are therefore ruled out.

According to (3.1) and (3.2) we are concerned with constructing asymptotic expansions as three independent parameters (ϵ, Ω and Re^{-1} or ϵ, K and Re^{-1}) simultaneously go to zero. Under such circumstances it is important to fix the relative orders of magnitude of the parameters. As far as the magnitude of ϵ is concerned, we shall mainly consider the case

$$\epsilon/K \rightarrow 0 \quad \text{as} \quad K \rightarrow 0. \tag{3.3}$$

This requires disturbance velocities that are even smaller than the small wavenumber. All perturbation equations will be linear in this case. We shall also

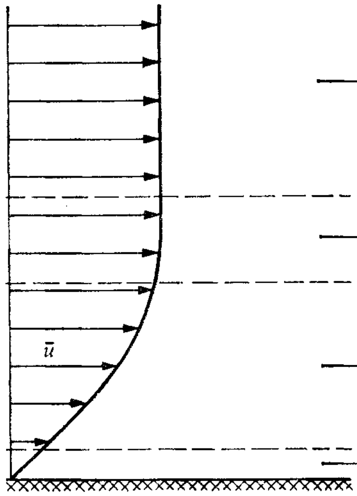

FIGURE 2. Division of the flow field with respect to the asymptotic expansions. Middle layer, shearing essential; inner layer, basic velocity small; outer layer, shearing unessential; transition layer, transition between outer and middle layer.

briefly investigate the more general case $\epsilon/K = O(1)$, which imposes weaker restrictions on the magnitude of the flow perturbations.

On the other hand, the relative order of magnitude of Ω and Re^{-1} will be established by defining a viscosity–frequency parameter N:

$$N = \Omega^{-2} Re^{-1}. \tag{3.4}$$

It will be seen that N characterizes the relative importance of viscosity effects and unsteady phenomena in the perturbed flow. With regard to the magnitude of N we shall consider all three possible cases, namely $N \to 0$, $N = O(1)$ and $N \to \infty$.

3.1. *The quasi-steady region of the flow field*

The unsteady terms in the conservation equations (2.2)–(2.4) tend to zero as $\Omega \to 0$. The convective terms, however, are $O(1)$, except in a very thin inner layer near the wall, where $\bar{u} \to 0$ as $y \to 0$ (see figure 2). As far as the first-order expansion in terms of Ω is concerned, the flow outside the inner layer, i.e. almost the whole flow field, can therefore be treated as if it were steady. The source of the unsteady behaviour of the flow is the inner layer, which will be treated in § 3.2.

The length scale of the tangential variations of the flow disturbances is given by the wavelength, which is comparable with K^{-1}. Hence we introduce a stretched tangential co-ordinate

$$X = Kx \tag{3.5}$$

and all expansions are to be understood to be with X fixed.

The basic shear flow is given by $u = \bar{u}(y)$, $v = 0$, $p = p_\infty =$ constant, $\rho = \bar{\rho}(y)$, $T = \bar{T}(y)$ etc., where all quantities except the pressure depend on the normal distance y from the wall. Primes on shear-flow quantities will denote differentiation with respect to y.

Well-known results of the steady theory (cf., for instance, Lighthill 1953; Stewartson 1964) indicate that the velocity components and the pressure in the middle layer (which is essentially the shear layer with the inner layer excluded, cf. figure 2) can be written as

$$u = \bar{u}(y) + \epsilon \bar{u}'(y)(M_\infty^2 - 1)^{\frac{1}{2}}[F(t,X) - G(t,X)], \tag{3.6}$$

$$v = -\epsilon K \bar{u}(y)(M_\infty^2 - 1)^{\frac{1}{2}}[F_X(t,X) - G_X(t,X)], \tag{3.7}$$

$$p = p_\infty - \epsilon K \rho_\infty [F_X(t,X) + G_X(t,X)], \tag{3.8}$$

where the functions F and G are related to the velocity perturbation potential ϕ far from the wall (in the outer, non-rotational flow) by

$$\phi = F(t,\xi) + G(t,\eta), \tag{3.9}$$

with

$$\xi, \eta = K[x \mp (M_\infty^2 - 1)^{\frac{1}{2}} y]. \tag{3.10}$$

We note that, with respect to the boundary layer, $F(t,\xi)$ represents outgoing waves, whereas $G(t,\eta)$ represents incoming waves. Depending on the particular problem, either $G(t,\eta)$ or a relation between $F(t,\xi)$ and $G(t,\eta)$ is prescribed.

More details of the expansions in the quasi-steady regions of the flow field are given in the appendix. Here we only note that matching of *all* perturbation quantities requires the insertion of a transitional layer between the middle and the outer layers (see figure 2).

3.2. *The inner layer*

In order to find a solution which is valid near the wall we define an inner coordinate

$$y_i = K^{-1} y. \tag{3.11}$$

We next introduce the asymptotic expansions

$$\left.\begin{aligned} u &= \bar{u}(Ky_i) + \epsilon u_i(X, y_i, t) + \dots, \\ v &= \epsilon K^2 v_i(X, y_i, t) + \dots, \\ p &= p_\infty + \epsilon K p_i(X, y_i, t) + \dots, \\ \rho &= \bar{\rho}(Ky_i) + \epsilon \rho_i(X, y_i, t) + \dots, \\ T &= \bar{T}(Ky_i) + \epsilon T_i(X, y_i, t) + \dots, \end{aligned}\right\} \tag{3.12}$$

with $y_i = O(1)$ kept fixed, and expand the basic shear flow near the wall as $K \rightarrow 0$ according to

$$\left.\begin{aligned} \bar{u}(Ky_i) &= Ky_i \bar{u}'(0) + \dots = Ky_i + \dots, \\ \bar{\rho}(Ky_i) &= 1 + Ky_i \bar{\rho}'(0) + \dots, \end{aligned}\right\} \tag{3.13}$$

etc. Equations (2.1)–(2.3) then yield

$$\partial u_i / \partial X + \partial v_i / \partial y_i = 0, \tag{3.14}$$

$$\frac{\Omega}{K^2}\frac{\partial u_i}{\partial t} + \left(y_i + \frac{\epsilon}{K} u_i\right)\frac{\partial u_i}{\partial X} + v_i\left(1 + \frac{\epsilon}{K}\frac{\partial u_i}{\partial y_i}\right) + \frac{\partial p_i}{\partial X} = \frac{1}{K^4 Re}\frac{\partial^2 u_i}{\partial y_i^2}, \tag{3.15}$$

$$\partial p_i / \partial y_i = 0. \tag{3.16}$$

If the disturbances are small such that $\epsilon/K \to 0$ equation (3.15) reduces to the linear equation

$$\frac{\Omega}{K^2}\frac{\partial u_i}{\partial t}+y_i\frac{\partial u_i}{\partial X}+v_i+\frac{\partial p_i}{\partial X}=\frac{1}{K^4 Re}\frac{\partial^2 u_i}{\partial y_i^2}. \tag{3.17}$$

The relation $\epsilon/K \to 0$ has already been postulated in (3.3). However, we may conclude from an inspection of (3.15) that in the more general case $\epsilon/K = O(1)$ the orders of magnitude of the flow quantities will not be changed. As far as orders of magnitude are concerned the results to be presented will therefore remain valid even if $\epsilon/K = O(1)$. The functional form of the solutions of (3.17) (in particular the harmonic wave solutions) and the numerical results will, of course, be valid only if $\epsilon/K \to 0$.

The form of (3.17) is governed by the orders of magnitude of Ω/K^2 and $1/K^4 Re$. We have to note, however, that only two of the three parameters Ω, K and Re can be chosen independently. The third one is determined by the solution of the problem. Therefore all cases possible in principle have to be pursued.

The two possibilities $\Omega/K^2 \to \infty$ or $1/K^4 Re \to \infty$ can immediately be ruled out because in either of these cases only those terms in (3.17) that have been entirely dropped in the middle layer would be retained. Hence matching between the inner and middle layers would be impossible.

We next consider the case $\Omega/K^2 = O(1)$ or, without loss of generality, $\Omega/K^2 = 1$. The factor $1/K^4 Re$ can then be replaced by $1/\Omega^2 Re$, which has to be $O(1)$ or smaller.

Finally we study the case $\Omega/K^2 \to 0$, therefore dropping the unsteady term. Since $y_i = 0$ and $v_i = 0$ at the wall there must be a sublayer where the viscous term becomes important. It is therefore necessary to choose the co-ordinate stretching according to (3.11) such that the viscous term is retained. $K^4 Re = 1$, i.e. $K = Re^{-\frac{1}{4}}$, is such a choice. We also note that $1/\Omega^2 Re \to \infty$ in this case and only in this case.

Comparing the various cases we see that they are uniquely distinguished by the order of magnitude of the parameter $N = 1/\Omega^2 Re$, which has already been introduced in (3.4). Taking now the reduced frequency Ω and the Reynolds number Re as the basic parameters we have to specify the parameter K, which characterizes the order of magnitude of the wavenumber, as follows:

$$K=\begin{cases}\Omega^{\frac{1}{2}} & \text{if} \quad N\to 0 \quad \text{or} \quad N=O(1),\\ Re^{-\frac{1}{4}} & \text{if} \quad N\to\infty.\end{cases} \tag{3.18}$$

Accordingly (3.17) can be written as

$$j\frac{\partial u_i}{\partial t}+y_i\frac{\partial u_i}{\partial X}+v_i+\frac{\partial p_i}{\partial X}=N^j\frac{\partial^2 u_i}{\partial y_i^2}, \tag{3.19}$$

where

$$j=\begin{cases}1 & \text{if} \quad N\to 0 \quad \text{or} \quad N=O(1),\\ 0 & \text{if} \quad N\to\infty.\end{cases} \tag{3.20}$$

The system of equations now consists of the continuity equation (3.14), the normal momentum equation (3.16) and the tangential momentum equation

		Thickness	Viscous effects	Unsteady effects	Compressibility effects	Pressure gradient normal to wall
Outer layer		$\to \infty$	No	No	Yes	Yes
Transition layer		$O(\delta)$	No	No	No	No
Middle layer		$O(\delta)$	No	No	No	No
Inner layer	$N \to 0$	$O(\delta\Omega^{\frac{1}{2}})$	No	Yes	No	No
	$N = O(1)$	$O(\delta\Omega^{\frac{1}{2}}) = O(\delta Re^{-\frac{1}{3}})$	Yes	Yes	No	No
	$N \to \infty$	$O(\delta Re^{-\frac{1}{3}})$	Yes	No	No	No

TABLE 1. Main properties of the four layers of the disturbed flow field

(3.19). They contain only three unknown perturbation quantities, i.e. u_i, v_i and p_i. The energy equation is uncoupled and can be used to determine the second thermodynamic perturbation quantity, say ρ_i. Inspection of the system shows that the inner layer is quasi-incompressible with vanishing normal pressure gradient. Further characterization of the layer depends on the parameter N.

If $N \to \infty$ the flow disturbances in the inner layer may be considered as quasi-steady but viscous; if $N \to 0$ unsteady effects are essential but viscous effects are not; if $N = O(1)$ unsteady effects as well as viscosity play an important role in the disturbed flow.

After elimination of p_i and v_i from (3.14), (3.16) and (3.19) the following third-order equation for u_i is obtained:

$$j\frac{\partial^2 u_i}{\partial t\, \partial y_i} + y_i \frac{\partial^2 u_i}{\partial X\, \partial y_i} = N^j \frac{\partial^3 u_i}{\partial y_i^3}. \tag{3.21}$$

Boundary conditions are $u_i = 0$ and $v_i = 0$ at the wall. Using (3.19) we write

$$(u_i)_{y_i=0} = 0, \tag{3.22}$$

$$(\partial^2 u_i/\partial y_i^2)_{y_i=0} = -(\rho_\infty/N^j)\,[F_{XX}(t,X) + G_{XX}(t,X)] \quad \text{if} \quad N \neq 0 \tag{3.23a}$$

and

$$(\partial u_i/\partial t)_{y_i=0} = \rho_\infty[F_{XX}(t,X) + G_{XX}(t,X)] \quad \text{if} \quad N = 0. \tag{3.23b}$$

Further boundary conditions follow from the requirement that all flow quantities match in an overlap domain between the inner layer and the adjoining middle layer. Comparison of (3.12) with (3.6) and (3.7) yields the condition

$$(\partial u_i/\partial X)_{y_i\to\infty} = (M_\infty^2 - 1)^{\frac{1}{2}}\,[F_X(t,X) - G_X(t,X)]. \tag{3.24}$$

Furthermore we must, of course, require that u_i be bounded as $y_i \to \infty$.

To conclude the section the main properties of the four layers are summarized in table 1. Compressibility effects and pressure gradients normal to the wall are seen to be important in the outer layer only, while unsteady effects and viscosity control the flow disturbances in the inner layer. It is this latter layer which allows unsteady disturbances to propagate far upstream. The middle layer as well as the transition layer, i.e. almost the whole boundary-layer region, act only as kind of a buffer between the inner and outer layers. This is why the shapes of the velocity and temperature profiles in the boundary layer do not influence the result.

4. Harmonic waves at a single wall

Equation (3.21) permits solutions in the form of progressive harmonic waves:

$$u_i = \alpha(y_i)\, e^{i(t+kX)}, \tag{4.1}$$

where k is the complex wavenumber with respect to the stretched tangential co-ordinate X. According to (3.5) the wavenumber with respect to the original co-ordinate x (based on the boundary-layer thickness) is therefore equal to Kk, with $K \to 0$ as the reduced frequency $\Omega \to 0$. Let us also recall that the time t is based on the reciprocal of the frequency.

Thus we may rewrite (4.1) as

$$u_i = \alpha(y_i)\, e^{\beta x}\, e^{i(t+2\pi x/\lambda)}, \tag{4.2}$$

where the damping exponent β and the wavelength λ are defined by the relations

$$\beta = -K\,\mathrm{Im}\,(k), \quad \lambda = 2\pi/[K\,\mathrm{Re}\,(k)]. \tag{4.3}$$

Note that for upstream-propagating damped waves, which are the object of our interest, λ and β have to be positive; hence $\mathrm{Re}\,(k) > 0$ and $\mathrm{Im}\,(k) < 0$.

4.1. *Solutions for finite values of the viscosity–frequency parameter N*

We first consider the case of finite N, i.e. $j = 1$. Substituting for u_i in (3.21) according to (4.1) yields the following ordinary differential equation for the amplitude function $\alpha(y_i)$:

$$N\alpha''' - i(1+ky_i)\,\alpha' = 0. \tag{4.4}$$

The general solution of this equation can be expressed in terms of the Airy functions Ai and Bi (cf. Abramowitz & Stegun 1965, p. 446). Eliminating solutions that are unbounded as $y_i \to \infty$ and using the boundary condition (3.22) we obtain

$$\alpha(y_i) = C\int_0^{y_i} \mathrm{Ai}\,(z)\, dy_i, \tag{4.5}$$

where C is an arbitrary complex constant (amplitude) and

$$z = z_1(1+ky_i), \tag{4.6}$$

with the complex constant z_1 uniquely defined by

$$|z_1| = |Nk^2|^{-\frac{1}{3}}, \quad \arg(z_1) = \tfrac{1}{6}\pi - \tfrac{2}{3}\arg(k). \tag{4.7}$$

In the boundary conditions (3.24) and (3.23*a*) we now focus our attention on a single wall with no waves propagating towards the wall from outside the boundary layer. Hence $G(t, X) \equiv 0$ and (3.24) and (3.23*a*) can be combined to yield the boundary condition

$$\left(\frac{\partial^2 u_i}{\partial y_i^2}\right)_{y_i=0} = -\frac{\rho_\infty}{N(M_\infty^2-1)^{\frac{1}{2}}}\left(\frac{\partial^2 u_i}{\partial X^2}\right)_{y_i\to\infty}, \tag{4.8}$$

where, according to the perfect-gas relation, ρ_∞ may be replaced by T_∞^{-1}, i.e. the ratio of the wall temperature and the free-stream temperature.

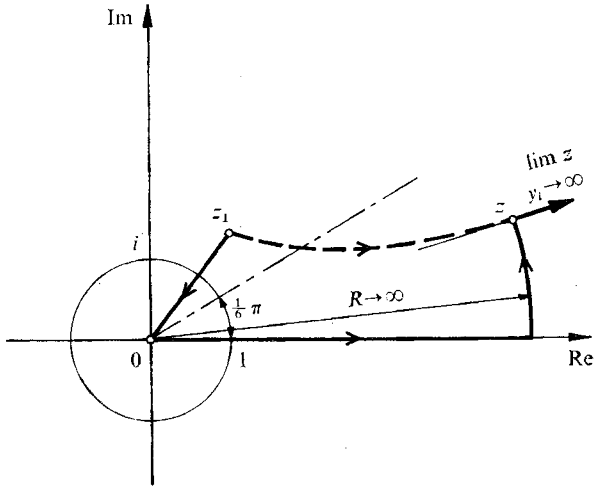

FIGURE 3. Path of integration in the complex plane for deducing (4.11) from (4.10).

Applying (4.8) to the solution represented by (4.1) and (4.3) we find that the wavenumber k has to satisfy the relation

$$NT_\infty(M_\infty^2-1)^{\frac{1}{2}}-Z(k)=0, \tag{4.9}$$

with
$$Z(k)=\frac{1}{z_1^2\mathrm{Ai}'(z_1)}\lim_{y_i\to\infty}\int_{z_1}^{z(y_i)}\mathrm{Ai}(\bar{z})\,d\bar{z}. \tag{4.10}$$

The integral in (4.10) can be expressed in a more convenient form by taking the path of integration indicated by the solid line in figure 3 and using the known asymptotic properties of the Airy functions. The result is

$$Z(k)=\frac{1}{z_1^2\mathrm{Ai}'(z_1)}\left[\frac{1}{3}-\int_0^{z_1}\mathrm{Ai}\,(z)\,dz\right]. \tag{4.11}$$

Equation (4.9), together with (4.10) or (4.11), represents a dispersion relation for the harmonic waves propagating upstream. By solving (4.9), k is obtained in terms of the viscosity–frequency parameter N, the free-stream temperature T_∞ (based on the wall temperature) and the free-stream Mach number M_∞.

The reduced frequency Ω does not appear explicitly in (4.9) but nevertheless the result depends on the frequency in a twofold way. On the one hand the parameter N contains Ω as can be seen from (3.4). On the other hand, the wavenumber with respect to the unstretched co-ordinate x is given by Kk, where K is also related to Ω, according to (3.18).

Numerical results for k will be presented in § 6. Anticipating, however, that the real and imaginary parts of k are $O(1)$ if $N=O(1)$, a result which is quite obvious from (4.9), we can already draw some remarkable conclusions concerning orders of magnitudes.

(i) The solution (4.1) represents upstream-propagating waves with very small reduced frequencies ($\Omega\to 0$).

(ii) The damping exponent β is proportional to K, i.e. comparable with $\Omega^{\frac{1}{2}}$.

	$N \to 0$	$N = O(1)$	$N \to \infty$
Damping exponent, β	$O(\Omega^{\frac{1}{2}})$	$O(\Omega^{\frac{1}{2}}) = O(Re^{-\frac{1}{4}})$	$O(Re^{-\frac{1}{4}})$†
Wavelength, λ	$O(\Omega^{-\frac{1}{2}})$	$O(\Omega^{-\frac{1}{2}}) = O(Re^{\frac{1}{4}})$	$O(Re^{\frac{1}{4}}N^{\frac{1}{2}})$
Phase velocity	$O(\Omega^{\frac{1}{2}})$	$O(\Omega^{\frac{1}{2}}) = O(Re^{-\frac{1}{4}})$	$O(Re^{-\frac{1}{4}}N^{-\frac{1}{2}})$

† Agrees with Lighthill's (1953) and Müller's (1953, 1955) results.

TABLE 2. Orders of magnitude of damping exponent β, wavelength λ and phase velocity for waves propagating upstream in the supersonic boundary layer at a plane wall. Small reduced frequencies Ω and large Reynolds numbers Re. (Lengths based on boundary-layer thickness, phase velocity on the sound velocity.)

This means that the waves under consideration are only weakly damped over path lengths of the order of the boundary-layer thickness.

(iii) The wavelength is comparable with $\Omega^{-\frac{1}{2}}$, i.e. very large, when scaled with the boundary-layer thickness.

(iv) The phase velocity (as well as the group velocity) is comparable with $\Omega^{\frac{1}{2}}$ when scaled with the free-stream velocity, i.e. the phase velocity (as well as the group velocity) is very small in comparison with the sound velocity.

4.2. *The case of weak viscosity effects* ($N \to 0$)

Expanding the wavenumber k for small viscosity–frequency parameters, $N \to 0$, according to the asymptotic series

$$k = k^{(0)} + N^{\frac{1}{2}}k^{(1)} + \dots \quad (\text{for } N \to 0), \tag{4.12}$$

we obtain from (4.9) and (4.10)

$$k^{(0)} = 2^{-\frac{1}{2}}T_{\infty}^{\frac{1}{2}}(M_{\infty}^2 - 1)^{\frac{1}{4}}(1 - i), \tag{4.13}$$

$$k^{(1)} = -2^{-\frac{3}{2}}T_{\infty}(M_{\infty}^2 - 1)^{\frac{1}{2}}(1 + i). \tag{4.14}$$

The first-order result (4.13) can also be obtained by direct solution of (3.21) with $N = 0$ (inviscid waves). The sign of the second-order term (4.14) indicates that owing to weak viscosity effects the damping of the waves as well as their wavelength is increased relative to the inviscid limit.

4.3. *The quasi-steady limit* ($N \to \infty$)

The limiting case of very large values of N (i.e. viscosity effects dominant relative to unsteady effects) can be investigated either by studying the behaviour of the solution obtained in § 4.1 as $N \to \infty$, or by solving (3.21) for $j = 0$. Both methods lead to the same result if we take into account the different magnitude of the stretching factor K in the two cases, cf. equation (3.18).

Irrespective of which description we prefer, the result can be written as

$$Kk = -\beta i + \dots \quad (\text{for } N \to \infty), \tag{4.15}$$

with

$$\beta = (3c_2T_{\infty})^{\frac{3}{4}}(M_{\infty}^2 - 1)^{\frac{3}{8}}Re^{-\frac{1}{4}}, \tag{4.16}$$

where $c_2 = -\mathrm{Ai}'(0) = 0{\cdot}2588\dots$. Recall that Kk is the wavenumber with respect to the unstretched co-ordinate x, and note that the real part of Kk vanishes in the limit $N \to \infty$. Hence, in agreement with what follows from (3.21) for $j = 0$,

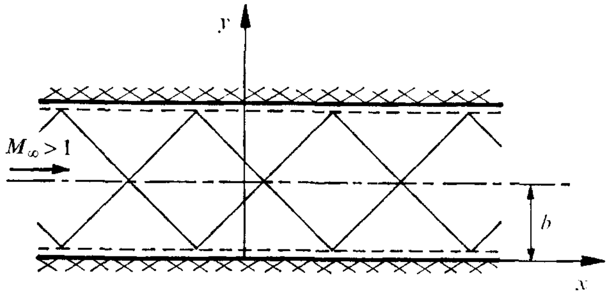

FIGURE 4. Upstream wave propagation in a plane-walled supersonic channel.

the limit $N \to \infty$ leads to quasi-steady behaviour of the upstream-propagating disturbances. Comparison of (4.16) with the result obtained by Lighthill (1953) (cf. also Stewartson 1964) for the logarithmic decay of steady disturbances of a supersonic boundary-layer flow shows complete agreement.

The orders of magnitude of the damping exponent β, the wavelength λ and the phase velocity of the upstream-propagating waves at a single wall are summarized in table 2.

5. Harmonic waves in a channel

So far we have looked for solutions for waves propagating upstream in a supersonic boundary layer at a single wall, with the flowing gas extending to infinity in the direction normal to the wall. We shall now turn our attention to the laterally bounded problem of supersonic flow between two plane parallel plates (see figure 4).

As far as upstream-propagating waves are concerned there is the following essential difference between the unbounded and the bounded problems. In the unbounded case the disturbances that leave the boundary layer are transferred along characteristics downstream to infinity. In the bounded case, however, the disturbances leaving the boundary layer and propagating downstream eventually reach the boundary layer at the opposite wall. After some interaction with the boundary layer, the disturbances are reflected towards the wall where they came from. This mechanism provides a possibility of conserving, in the absence of viscosity, the energy of upstream-propagating waves. Hence we shall find undamped waves in the bounded problem if viscosity, as far as the disturbances are concerned, is neglected ($N = 0$), whereas for the single wall the upstream-propagating waves are damped even in the inviscid case as we have already seen in § 4.2.

5.1. *Solutions for arbitrary values of the viscosity–frequency parameter N*

Progressive harmonic waves can be found as solutions of the laterally bounded problem in the same manner as has been done for the single wall in § 4. Equations (4.1)–(4.7) remain valid without any changes. Thus the solution is formally the same here as it was for the single wall, but the dispersion relation which determines the wavenumber k is different.

We seek solutions that are symmetrical about the centre-line of the channel. Hence the normal velocity perturbation in the outer region has to vanish at the centre-line, i.e.

$$\phi_y = 0 \quad \text{at} \quad y = b = b_0/K, \tag{5.1}$$

where b is the half-width of the channel in the unstretched co-ordinate system (x,y), i.e. b is made dimensionless by the boundary-layer thickness δ, whereas $b_0 = Kb$ can be considered as a reduced half-width which is based on a length of the order of the wavelength. With (3.9) the condition (5.1) can be written as a relation between the wave functions $F(t,\xi)$ and $G(t,\eta)$ as follows:

$$F_\xi(t, X - b_0(M_\infty^2-1)^{\frac{1}{2}}) = G_\eta(t, X + b_0(M_\infty^2-1)^{\frac{1}{2}}). \tag{5.2}$$

Now introducing harmonic waves as in (4.1) we obtain from the boundary conditions (3.24) and (3.23a) together with the symmetry condition (5.2) the following dispersion relation for k:

$$\frac{NT_\infty(M_\infty^2-1)^{\frac{1}{2}} - Z(k)}{NT_\infty(M_\infty^2-1)^{\frac{1}{2}} + Z(k)} = \exp\left[-2b_0(M_\infty^2-1)^{\frac{1}{2}} ki\right]. \tag{5.3}$$

Here $Z(k)$ is, as before, given by (4.10) or alternatively by (4.11).

Numerical results obtained by evaluation of (5.3) will be given in § 6.

If $b_0 \to \infty$ equation (5.3) reduces, as it should, to equation (4.9) for the single wall, provided that the imaginary part of k is finite and negative.

5.2. *The inviscid case* ($N = 0$) *and the limit* $N \to 0$

For small values of the viscosity–frequency parameter N we expand the wavenumber k as

$$k = k^{(0)} + N^{\frac{1}{2}}k^{(1)} + \ldots \quad (N \to 0). \tag{5.4}$$

Introducing this expansion into the dispersion relation (5.3) and assuming b_0 to be $O(1)$ we obtain for the first- and second-order terms the equations

$$\frac{T_\infty(M_\infty^2-1)^{\frac{1}{2}} - k^{(0)2}i}{T_\infty(M_\infty^2-1)^{\frac{1}{2}} + k^{(0)2}i} = \exp\left[-2b_0(M_\infty^2-1)^{\frac{1}{2}} k^{(0)}i\right], \tag{5.5}$$

$$k^{(1)} = (1-i)\,2^{-\frac{1}{2}}T_\infty k^{(0)3}\{2T_\infty k^{(0)} - b_0[T_\infty^2(M_\infty^2-1) + k^{(0)4}]\}^{-1}. \tag{5.6}$$

Equation (5.5) cannot be explicitly solved for $k^{(0)}$, but it can be seen that only a real $k^{(0)}$ can satisfy the equation. Furthermore, for upstream-propagating waves $k^{(0)}$ has to be positive. Using this knowledge about $k^{(0)}$ equation (5.5) can be solved for b_0 to yield

$$b_0 = k^{(0)-1}(M_\infty^2-1)^{-\frac{1}{2}}\{\arctan[k^{(0)2}T_\infty^{-1}(M_\infty^2-1)^{-\frac{1}{2}}] + \tfrac{1}{2}n\pi\} \quad (n = 0, 1, 2, \ldots). \tag{5.7}$$

A graphical representation of (5.7) is given in figure 5. Hence it follows that in a supersonic flow between two plane parallel walls, weak waves ($\epsilon \to 0$) with small reduced frequencies ($\Omega \to 0$) would propagate upstream without damping if viscosity were neglected; furthermore, there would be a countably infinite set of possible wavenumbers for any particular channel width.

Taking small viscosity effects into account, however, changes the results drastically. Figure 6 shows the imaginary part of $k^{(1)}$ as given by (5.6). It can

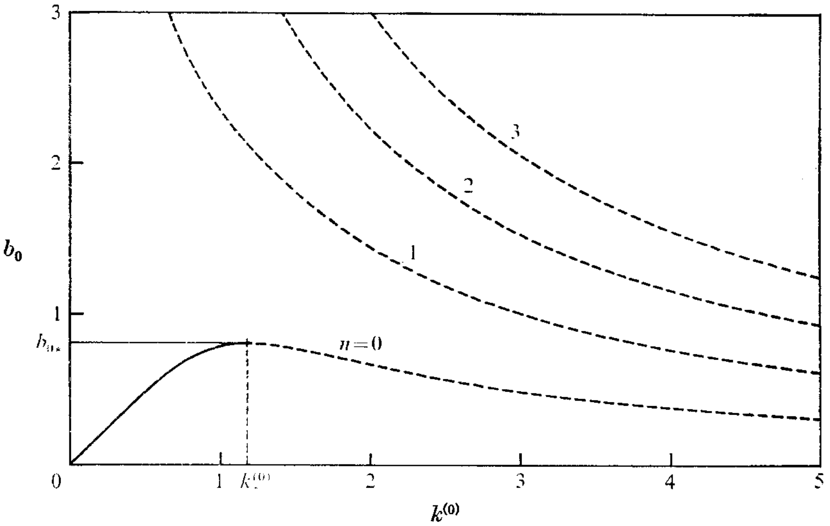

FIGURE 5. The inviscid wavenumber $k^{(0)}$ as a function of the stretched channel half-width b_0. Free-stream Mach number $M_\infty = \sqrt{2}$, free-stream-to-wall temperature ratio $T_\infty = 1$.

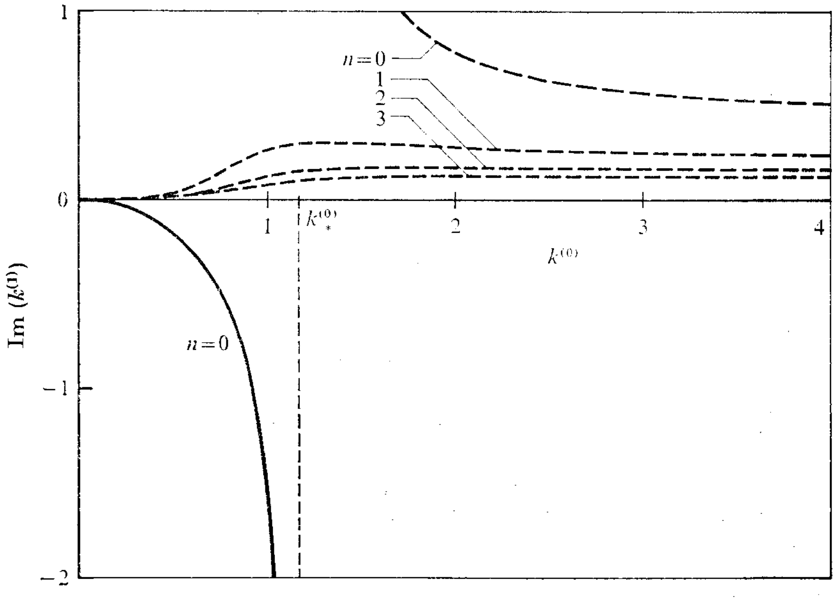

FIGURE 6. The imaginary part of the second-order wavenumber for weak viscosity effects ($N \rightarrow 0$). Free-stream Mach number $M_\infty = \sqrt{2}$, free-stream-to-wall temperature ratio $T_\infty = 1$.

be seen that only one branch of one solution ($n = 0$) has the correct negative sign, leading to damping of upstream-propagating waves. The other solutions have to be ruled out because positive imaginary parts of the wavenumber have been excluded in the course of the calculations. Moreover $k^{(1)}$ becomes infinite at a certain value of $k^{(0)}$, say $k^{(0)}_*$, indicating that the expansion (5.4) breaks down as $k^{(0)} \rightarrow k^{(0)}_*$. Going back now to figure 5 we note that in all solutions only the

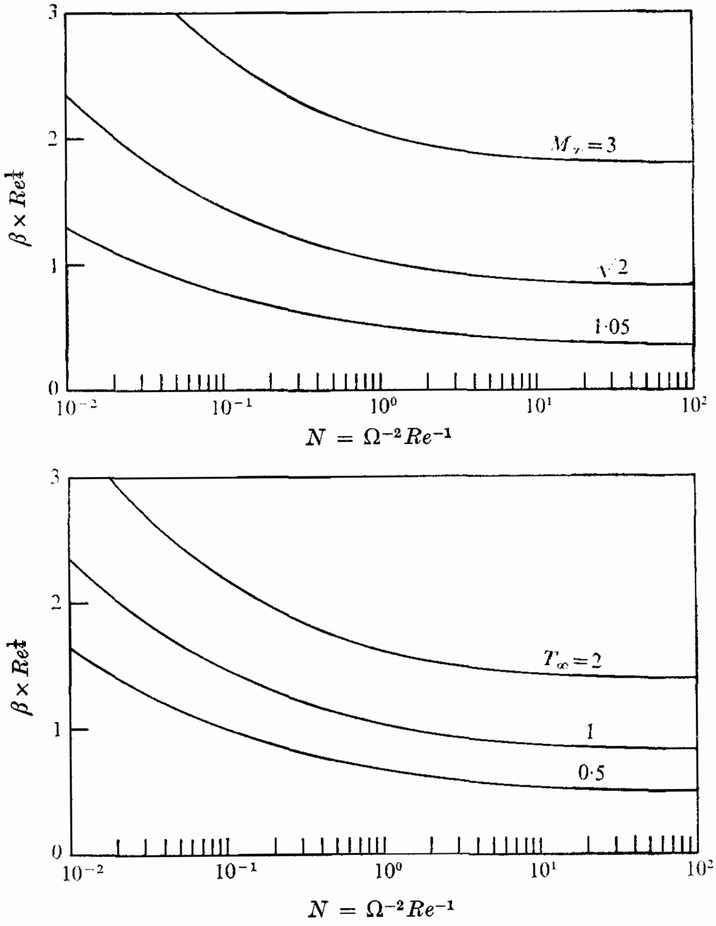

FIGURE 7. The damping exponent β for upstream-propagating waves near a single wall. (a) As a function of the viscosity–frequency parameter $N = \Omega^{-2}Re^{-1}$ and the free-stream Mach number M_∞. Free-stream temperature and wall temperature are equal, $T_\infty = 1$. (b) As a function of the viscosity–frequency parameter $N = \Omega^{-2}Re^{-1}$ and the free-stream temperature T_∞ (based on the wall temperature). Free-stream Mach number $M_\infty = \sqrt{2}$.

solid part of the $n = 0$ curve is correct from the point of view of a limit $N \to 0$. Since $b_0(k^{(0)})$ for $n = 0$ reaches its maximum value b_{0*} exactly at $k^{(0)} = k_*^{(0)}$ the expansion (5.4) and the results (5.7) and (5.6) that follow from it are useful only for half-widths below the critical value $b_* = b_{0*}/K$.

In closing this section the following remark seems to be appropriate. When we expanded the equations for small disturbances ($\epsilon \to 0$) we did not take into account the far-field effects due to perturbations of the characteristics. This could be done by the method of strained co-ordinates, for example. Those far-field effects will certainly cause weak damping of the waves in the inviscid case ($N = 0$) and may also be an essential source of damping for sufficiently small values of N. Since $\Omega^{\frac{1}{2}}$ has to be small, such extremely small values of N require extraordinarily large Reynolds numbers which appear to be beyond the regime which is of practical interest.

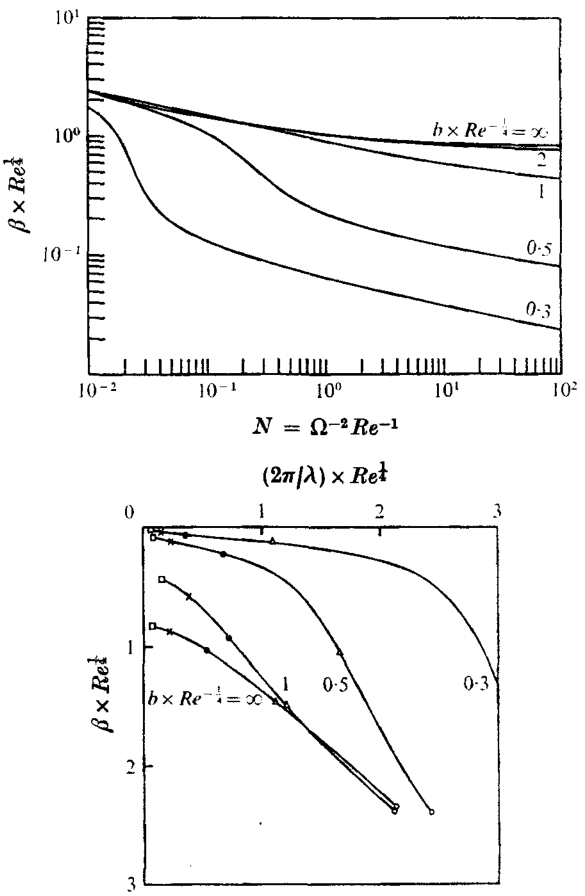

FIGURE 8. The damping exponent β for a channel. (*a*) As a function of the viscosity-frequency parameter $N = \Omega^{-2}Re^{-1}$ and the channel half-width b (based on the boundary-layer thickness). $M_\infty = \sqrt{2}, T_\infty = 1$ (free-stream temperature and wall temperature equal). (*b*) As a function of the wavelength λ and the channel half-width. Values of the viscosity-frequency parameter are indicated as follows: ○, $N = 0{\cdot}01$; △, $N = 0{\cdot}1$; ●, $N = 1$; ×, $N = 10$; □, $N = 100$. (All lengths based on the boundary-layer thickness.)

6. Numerical results

As part of the preceding solutions, equations for k, the complex wavenumber with respect to the stretched co-ordinate X, have been obtained. For a laterally unbounded gas stream along a single plane wall, k has to satisfy (4.9); for a flow between two plane parallel walls (5.3) applies instead. Both equations have been solved numerically with various values of the parameters N, M_∞, T_∞ and b_0. The damping exponent β and the wavelength λ have then been computed from (4.3). Note that the shapes of the velocity and temperature profiles in the boundary layer are left unspecified as they do not affect the results. Some results are presented in figures 7 and 8. Since under experimental conditions the reduced frequency Ω is likely to be more easily varied than the Reynolds number Re, all quantities are now based on powers of Re rather than Ω.

For a single wall, the damping exponent β *vs.* the viscosity–frequency parameter N has been plotted in figure 7, figure 7 (*a*) showing the influence of the free-stream Mach number M_∞ and figure 7 (*b*) showing the effect of changing the ratio of the free-stream temperature and wall temperature.

Results for the upstream propagation in a channel are presented in figure 8. Figure 8 (*a*) shows how the damping exponent depends on the viscosity–frequency parameter N and the channel half-width b. Figure 8 (*b*) is a plot of the wavenumbers in the complex plane, giving us the damping exponent as a function of the reciprocal of the wavelength.

When the Reynolds number Re of the basic flow and the channel half-width b are kept constant, the damping exponent β is seen to increase with increasing reduced frequency Ω. As far as the influence of the channel width is concerned, there is a general trend of decreasing damping with decreasing channel width; in a certain range of N and b, however, the general trend is reversed, but quantitatively this inverse effect is quite small.

In order to indicate how Lighthill's (1953) theory is incorporated in the present investigation it should be mentioned that Lighthill's theory provides the right-hand asymptote to the curve labelled $b\,Re^{-\frac{1}{4}} = \infty$ in our figure 8 (*a*).

The author is grateful to Professor K. Oswatitsch for stimulating interest in the subject. The author has also benefited from several discussions with Dr W. Koch and wishes to thank Mr L. Leopold for the computer programming. A referee's comment led to a more concise representation.

Appendix. Matched asymptotic expansions for the quasi-steady layers

Middle layer

The asymptotic expansions are

$$\left.\begin{aligned} u &= \overline{u}(y) + \epsilon u_m(X,y,t) + \ldots, \\ v &= \epsilon K v_m(X,y,t) + \ldots, \\ p &= p_\infty + \epsilon K p_m(X,y,t) + \ldots, \\ \rho &= \overline{\rho}(y) + \epsilon \rho_m(X,y,t) + \ldots, \\ T &= \overline{T}(y) + \epsilon T_m(X,y,t) + \ldots. \end{aligned}\right\} \tag{A 1}$$

Using the relation $K\,Re \to \infty$ as $Re \to \infty$, which means that the wavelength of the disturbances is small relative to the spatial extent of the boundary layer in upstream direction, we obtain from (2.1)–(2.4) the following equations for the first-order perturbation quantities:

$$\partial u_m/\partial X + \partial v_m/\partial y = 0, \tag{A 2}$$

$$\overline{u}\,\partial u_m/\partial X + \overline{u}' v_m = 0, \tag{A 3}$$

$$\partial p_m/\partial y = 0, \tag{A 4}$$

$$\overline{u}\,\partial \rho_m/\partial X + \overline{\rho}' v_m = 0. \tag{A 5}$$

The general solution of the system (A 2)–(A 5) is

$$u_m = f(t,X)\,\bar{u}'(y) + g(t,y), \tag{A 6}$$

$$v_m = -f_X(t,X)\,\bar{u}(y), \tag{A 7}$$

$$p_m = p_m(t,X), \tag{A 8}$$

$$\rho_m = f(t,X)\,\bar{\rho}'(y) + h(t,y), \tag{A 9}$$

where the functions f, g, h and p_m are to be determined from matching conditions.

Outer layer

Very far from the wall the flow field is described in terms of the outer co-ordinate

$$y_o = Ky, \tag{A 10}$$

where $y_o = O(1)$ as $K \to 0$.

As $y \to \infty$ the basic boundary-layer profiles differ from the inviscid free stream only by exponentially small terms. Hence the first two terms of the outer expansion can be written as

$$\left.\begin{aligned} u &= 1 + \epsilon K u_o(X, y_o, t) + \ldots, \\ v &= \epsilon K v_o(X, y_o, t) + \ldots, \\ p &= p_\infty + \epsilon K p_o(X, y_o, t) + \ldots, \\ \rho &= \rho_\infty + \epsilon K \rho_o(X, y_o, t) + \ldots . \end{aligned}\right\} \tag{A 11}$$

Here ρ_∞ stands for $\bar{\rho}(\infty)$, i.e. the (dimensionless) free-stream density.

Expanding the governing equations (2.1)–(2.4) according to (A 11) with X and y_o fixed we obtain the following system of perturbation equations:

$$\frac{1}{\rho_\infty}\frac{\partial \rho_o}{\partial X} + \frac{\partial u_o}{\partial X} + \frac{\partial v_o}{\partial y_o} = 0, \tag{A 12}$$

$$\rho_\infty\,\partial u_o/\partial X + \partial p_o/\partial X = 0, \tag{A 13}$$

$$\rho_\infty\,\partial v_o/\partial X + \partial p_o/\partial y_o = 0, \tag{A 14}$$

$$\frac{1}{p_\infty}\frac{\partial p_o}{\partial X} - \frac{\gamma}{\rho_\infty}\frac{\partial \rho_o}{\partial X} = 0. \tag{A 15}$$

Equations (A 13) and (A 15) can be integrated at once to yield, together with the condition that all disturbances vanish simultaneously at infinity, the relations

$$p_o = -\rho_\infty u_o, \quad \rho_o = M_\infty^2 p_o, \tag{A 16}$$

where, owing to our dimensionless notation, $\rho_\infty/\gamma p_\infty$ has been replaced by the square of the free-stream Mach number M_∞. Upon introduction of a velocity potential ϕ according to

$$\partial\phi/\partial X = u_o, \quad \partial\phi/\partial y_o = v_o \tag{A 17}$$

the remaining equations (A 12) and (A 14) reduce to the single equation

$$(M_\infty^2 - 1)\frac{\partial^2 \phi}{\partial X^2} - \frac{\partial^2 \phi}{\partial y_o^2} = 0. \tag{A 18}$$

This wave equation has the general solution (3.9).

Transition layer

This layer is characterized by the condition

$$\bar{u}'(y) = O(K), \tag{A 19}$$

which implies that $\bar{\rho}'(y) = O(K)$ or even smaller. Since the velocity in the boundary layer differs from the free-stream velocity, as $y \to \infty$, only by exponentially small terms, the y co-ordinate in the transition layer as defined by (A 19) becomes logarithmically large as $K \to 0$. The thickness of the layer, however, remains $O(1)$. Thus we introduce the normal co-ordinate y_{tr} in the transition layer by the relation

$$y = y^* + y_{tr}, \tag{A 20}$$

where the (logarithmically large) reference distance y^* characterizes the position of the transition layer. We may, for instance, define y^* such that $\bar{u}'(y^*) = K$.

Expanding (2.1) to (2.4) according to

$$\left.\begin{aligned} u &= \bar{u}(y) + \epsilon K u_{tr}(X, y_{tr}, t) + \ldots, \\ v &= \epsilon K v_{tr}(X, y_{tr}, t) + \ldots, \\ p &= p_\infty + \epsilon K p_{tr}(X, y_{tr}, t) + \ldots, \\ \rho &= \bar{\rho}(y) + \epsilon K \rho_{tr}(X, y_{tr}, t) + \ldots \end{aligned}\right\} \tag{A 21}$$

with y_{tr} fixed yields

$$\partial v_{tr}/\partial y_{tr} = 0, \tag{A 22}$$

$$\rho_\infty \left(\frac{\partial u_{tr}}{\partial X} + \frac{\bar{u}'}{K} v_{tr}\right) + \frac{\partial p_{tr}}{\partial X} = 0, \tag{A 23}$$

$$\partial p_{tr}/\partial y_{tr} = 0, \tag{A 24}$$

$$M_\infty^2 \frac{\partial p_{tr}}{\partial X} - \frac{\partial \rho_{tr}}{\partial X} - \frac{\bar{\rho}'}{K} v_{tr} = 0. \tag{A 25}$$

This system of equations has the solution

$$v_{tr} = v_{tr}(t, X), \tag{A 26}$$

$$p_{tr} = p_{tr}(t, X), \tag{A 27}$$

$$u_{tr} = -p_{tr}/\rho_\infty - \left[\int_0^X v_{tr}\, dX + \psi_1(t, y_{tr})\right] \bar{u}'/K, \tag{A 28}$$

$$\rho_{tr} = M_\infty^2 p_{tr} - \left[\int_0^X v_{tr}\, dX + \psi_2(t, y_{tr})\right] \bar{\rho}'/K, \tag{A 29}$$

where the functions v_{tr}, p_{tr}, ψ_1 and ψ_2 are to be determined from matching with the adjoining layers.

Matching

Applying one of the well-known matching rules (Van Dyke 1964) yields

$$p_m(t, X) = p_{tr}(t, X) = -\rho_\infty[F_X(t, X) + G_X(t, X)], \tag{A 30}$$

$$f_X(t,X) = -v_{tr}(t,X) = (M_\infty^2-1)^{\frac{1}{2}}[F_X(t,X)-G_X(t,X)], \tag{A 31}$$

$$g(t,y) \equiv h(t,y) \equiv 0, \tag{A 32}$$

$$\psi_1(t,y_{tr}) \equiv \psi_2(t,y_{tr}) \equiv 0. \tag{A 33}$$

Equations (3.6)–(3.8) follow from (A 6)–(A 8) and (A 30)–(A 32).

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